LECTURE 11
LINEAR PROGRAMMING

Instructor: Lu Wang

College of Business

Shanghai University of Finance and Economics

上海财经大学
Shanghai University of Finance and Economics

WHAT IS LINEAR PROGRAMMING (LP)?

- A process of transforming a real world problem into a mathematical model
- The most widely used modeling technique designed to help managers in planning and making decisions (IBM, Excel, Gurobi, etc.)
- A deterministic modeling technique
- Linear programming helps in resource allocation decisions (e.g., product mix, labour scheduling)

HISTORY OF LINEAR PROGRAMMING

- **First developed by Leonid Kantorovich in 1939.**
- **Was kept a secret for 8 years during WWII until George B. Dantzig (1947) published the simplex method and John von Neumann developed the theory of duality as a linear optimization solution, and applied it in the field of game theory.**
- **Many industries use it in their daily planning (airlines, oil companies, farming, auto industry,…)**

STEPS IN DEVELOPING A LP MODEL

- **Problem Formulation**
- **Solution Techniques**
- **Interpretation of Results**

STEP 1: PROBLEM FORMULATION

There are three components to a LP:

1. Decision variables
2. Objective Function
   - Profit or Cost Parameters
   - Maximization/Minimization
3. Constraints
   - Constraint Parameters
   - Right-Hand-Side Constants

STEP 1: PROBLEM FORMULATION

The objective function and constraints are linear functions of the decision variables

Objective function:

Maximize or ***Minimize***

Constraints: “$\geq$” or “$\leq$” or “=”

EXAMPLE LP FORMULATION FLAIR FURNITURE

- **Two products: chairs and tables**
- **Decision: How many units of each product to produce each month?**
- **Objective: Maximize profit**
- **Constraints: Limited resources (e.g., labour hours available), management or market restrictions**

FLAIR FURNITURE CO. DATA

	Tables (per table)	Chairs (per chair)	Hours Available
Unit Profit	$7	$5	
Carpentry	3 hrs	4 hrs	2400
Painting	2 hrs	1 hr	1000

Other Management/Market Limitations:

- Make no more than 450 chairs
- Make at least 100 tables

EXAMPLE LP FORMULATION: FLAIR FURNITURE

Decision Variables:

T = Number of tables to make each month

C = Number of chairs to make each month

Objective Function (OF): Maximize total profit

Objective: ***Maximize*** **Z = 7T + 5C** **(profit)**

Subject to:	**3T + 4C $\leq$ 2400**	**(carpentry hrs.)**
	2T + 1C $\leq$ 1000	**(painting hrs.)**
	C $\leq$ 450	**(max no. of chairs)**
	T $\geq$ 100	**(min no. of tables)**
	T $\geq$ 0, C $\geq$ 0	**(non-negativity)**

STEP 2: SOLUTION TECHNIQUES

There are many different ways to solve a linear program:

- Graphical Solutions
- Microsoft Excel (Solver)
- Simplex Method
- Karamarkar's Algorithm
- IBM CPLEX
- Gurobi
- MATLAB
- Mathematica

Example Graphic Solution: Flair Furniture

Carpentry

Constraint Line:

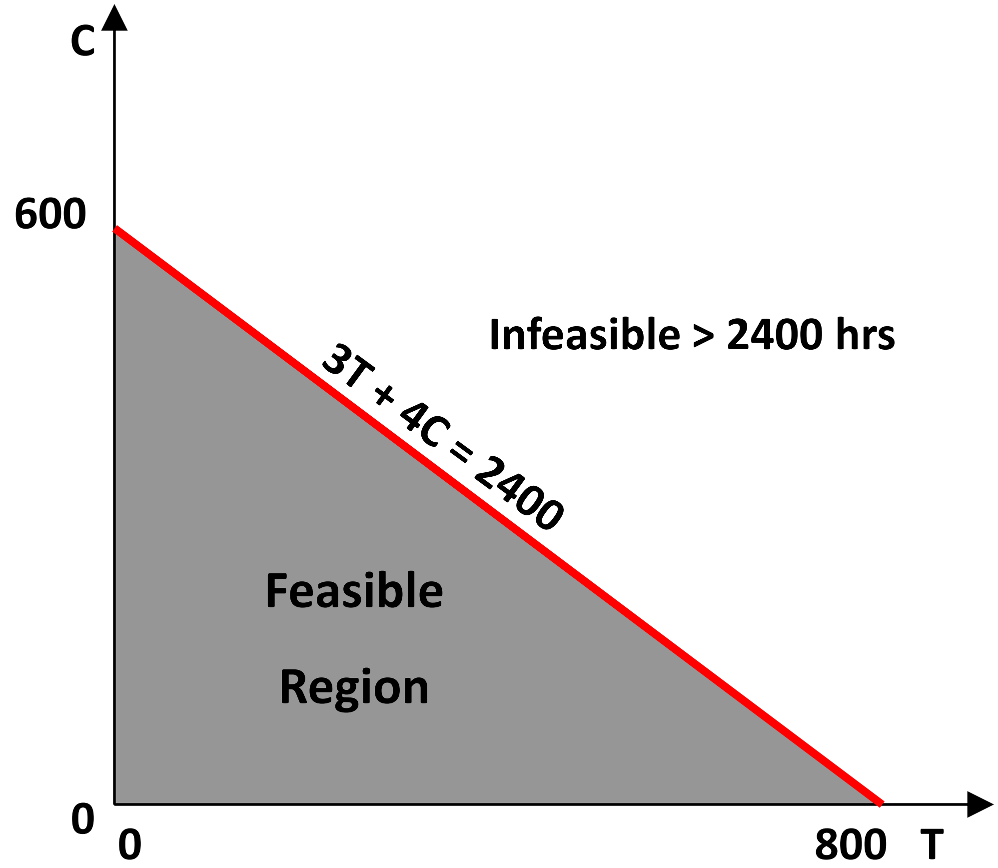
3T + 4C = 2400

(i.e., 3T + 4C ≤ 2400)

Intercepts:

(T = 0, C = 600)

(T = 800, C = 0)



Example Graphic Solution: Flair Furniture

<u>Painting</u>

Constraint Line:

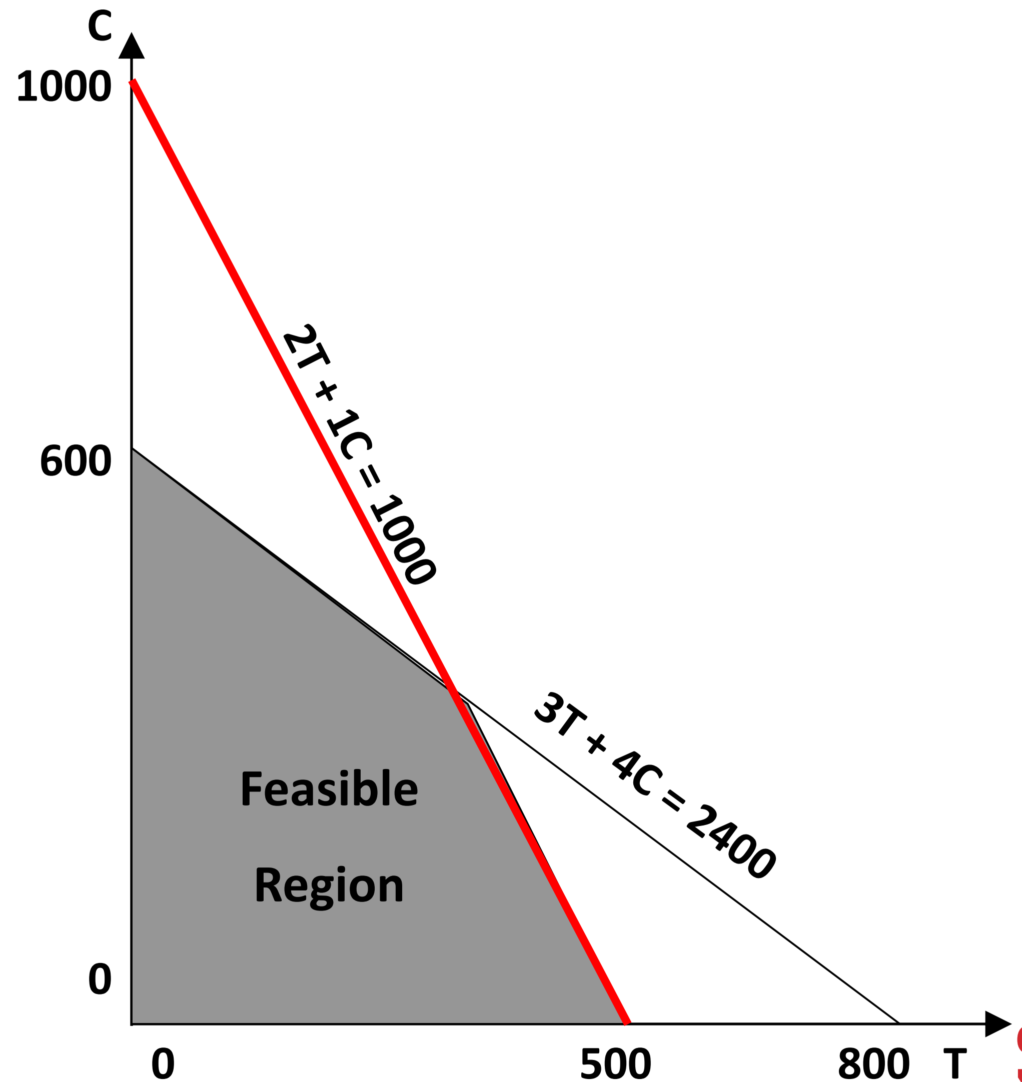
2T + 1C = 1000

(i.e., 2T + 1C ≤ 1000)

Intercepts:

(T = 0, C = 1000)

(T = 500, C = 0)



Example Graphic Solution: Flair Furniture

Max Chair Line:

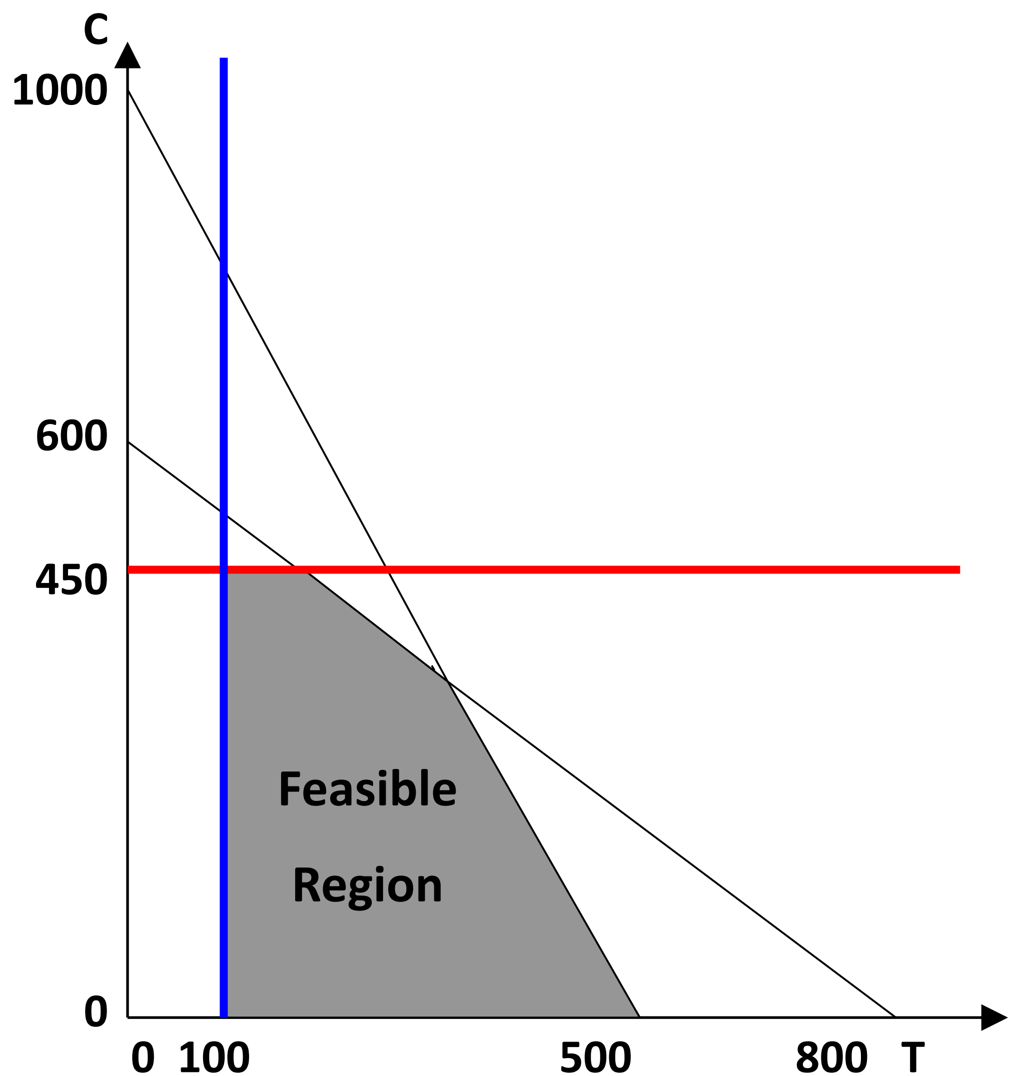
C = 450

(i.e., C $\leq$ 450)

Min Table Line:

T = 100

(i.e., T $\geq$ 100)



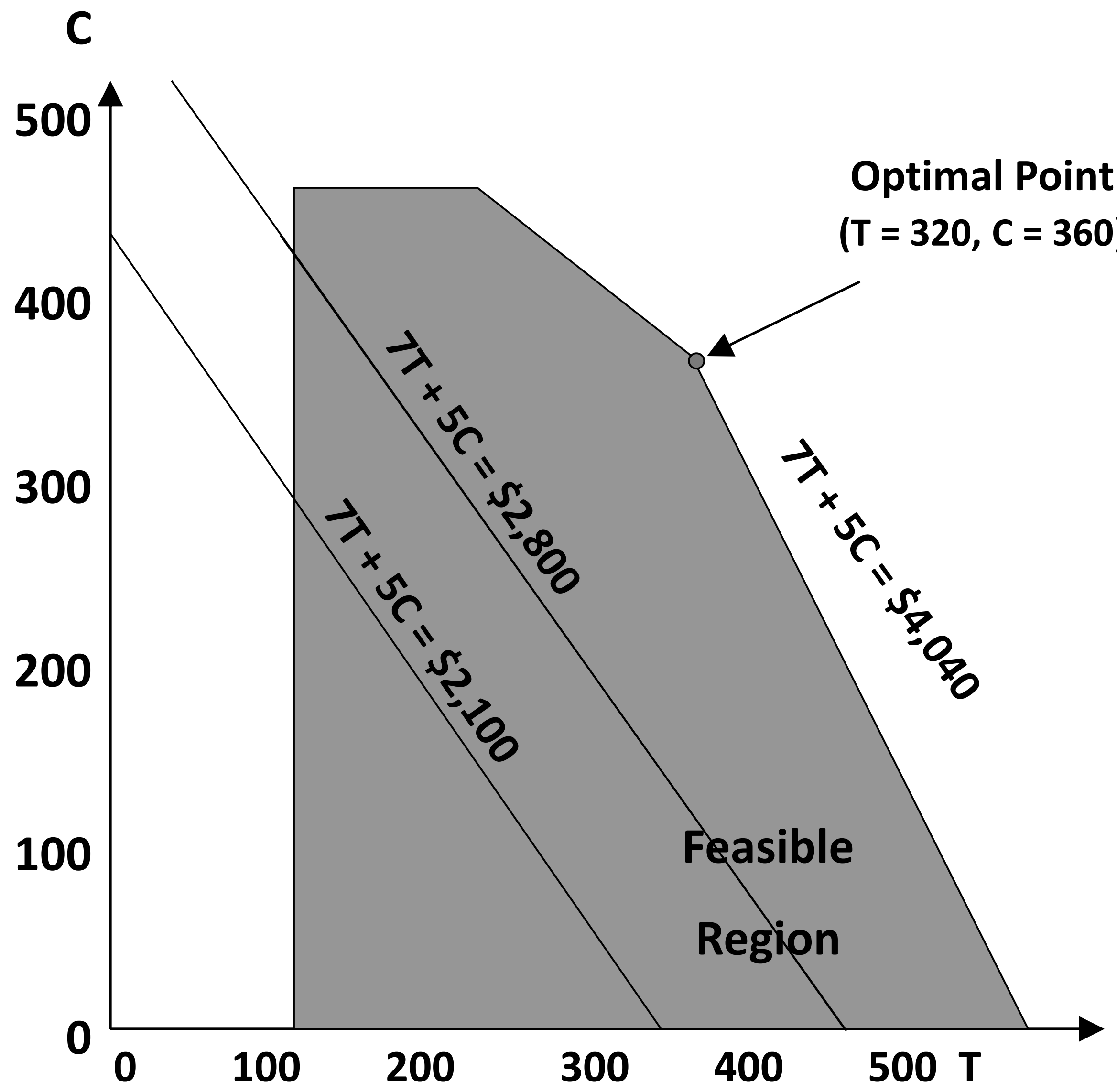
Example Graphic Solution: Flair Furniture

Objective Function Line:

Profit

=

7T + 5C



Example Graphic Solution: Flair Furniture

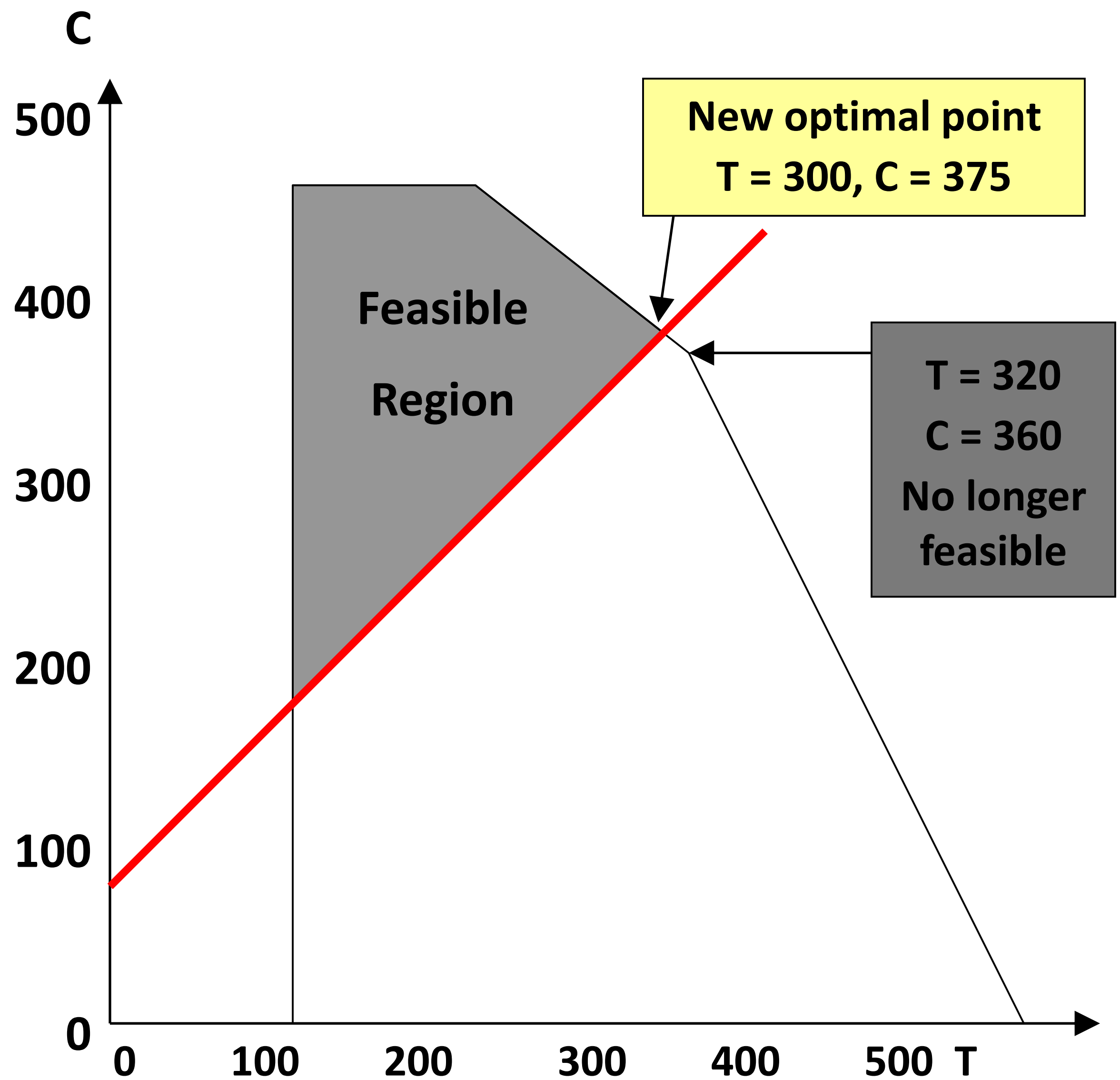
Additional Constraint:

Need at least 75 more chairs than tables

$C \geq T + 75$

or

$C - T \geq 75$



MAIN INSIGHTS AND DEFINITIONS

- **Feasible solution: A point that satisfies all the constraints in the problem.**
- **The set of all feasible solutions is called the feasible set or feasible region. It is always polygonal.**
- **Corner Point Property: An optimal solution to the LP must lie at one or more corner points.**
- **Optimal point (region): The corner point or region with the best objective function value**
  - (i.e., the largest for a maximization problem; the smallest for a minimization problem)

MAIN INSIGHTS AND DEFINITIONS

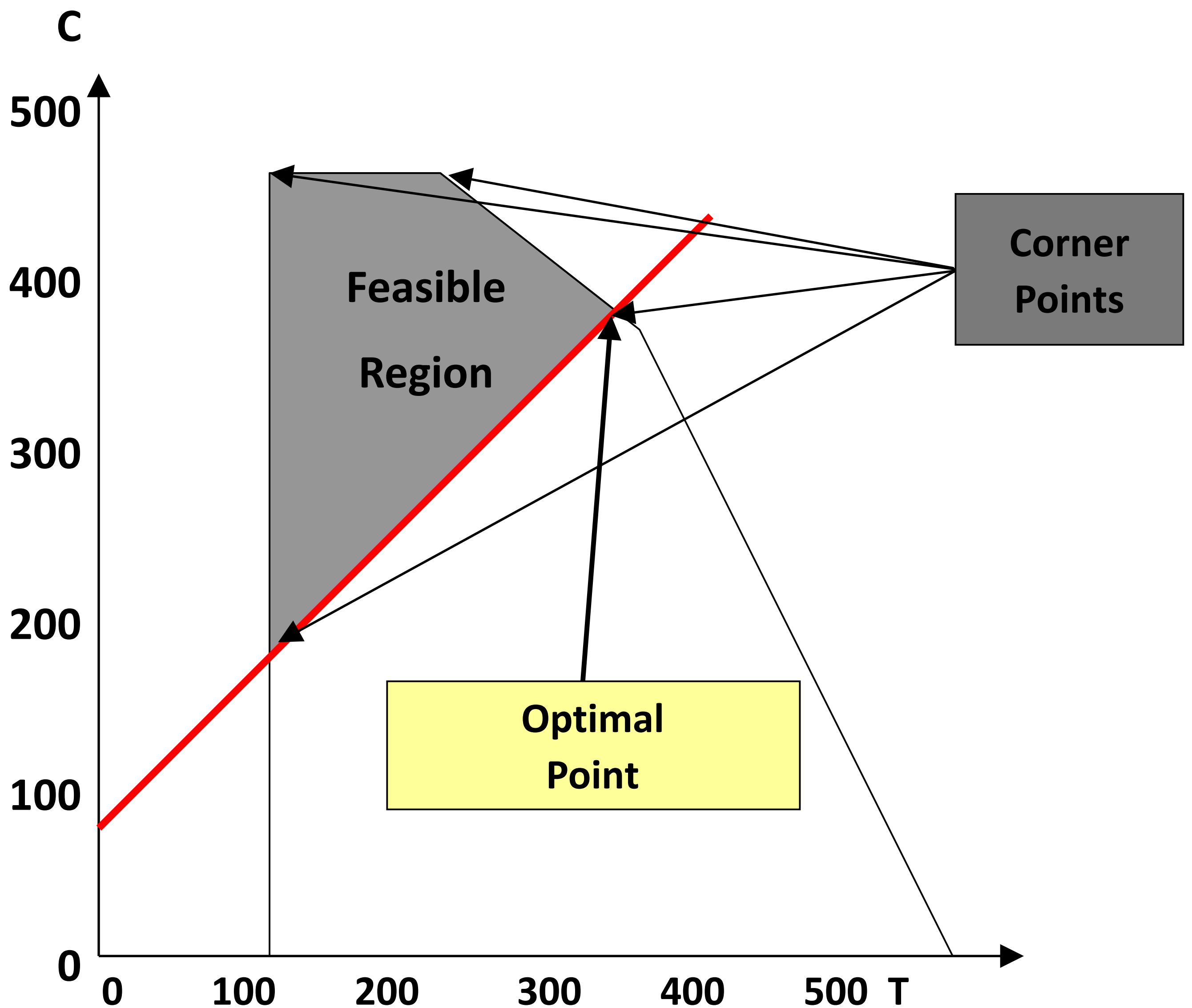
A constraint is said to be binding or active if it is satisfied with equality at the optimal solution.

- Geometrically, an active constraint is one that passes through the optimal solution.

A constraint is said to be inactive or nonbinding if it is satisfied with strict inequality at the optimal solution.

- Geometrically, an inactive constraint is one that does not pass through the optimal solution.

Example Graphic Solution: Flair Furniture

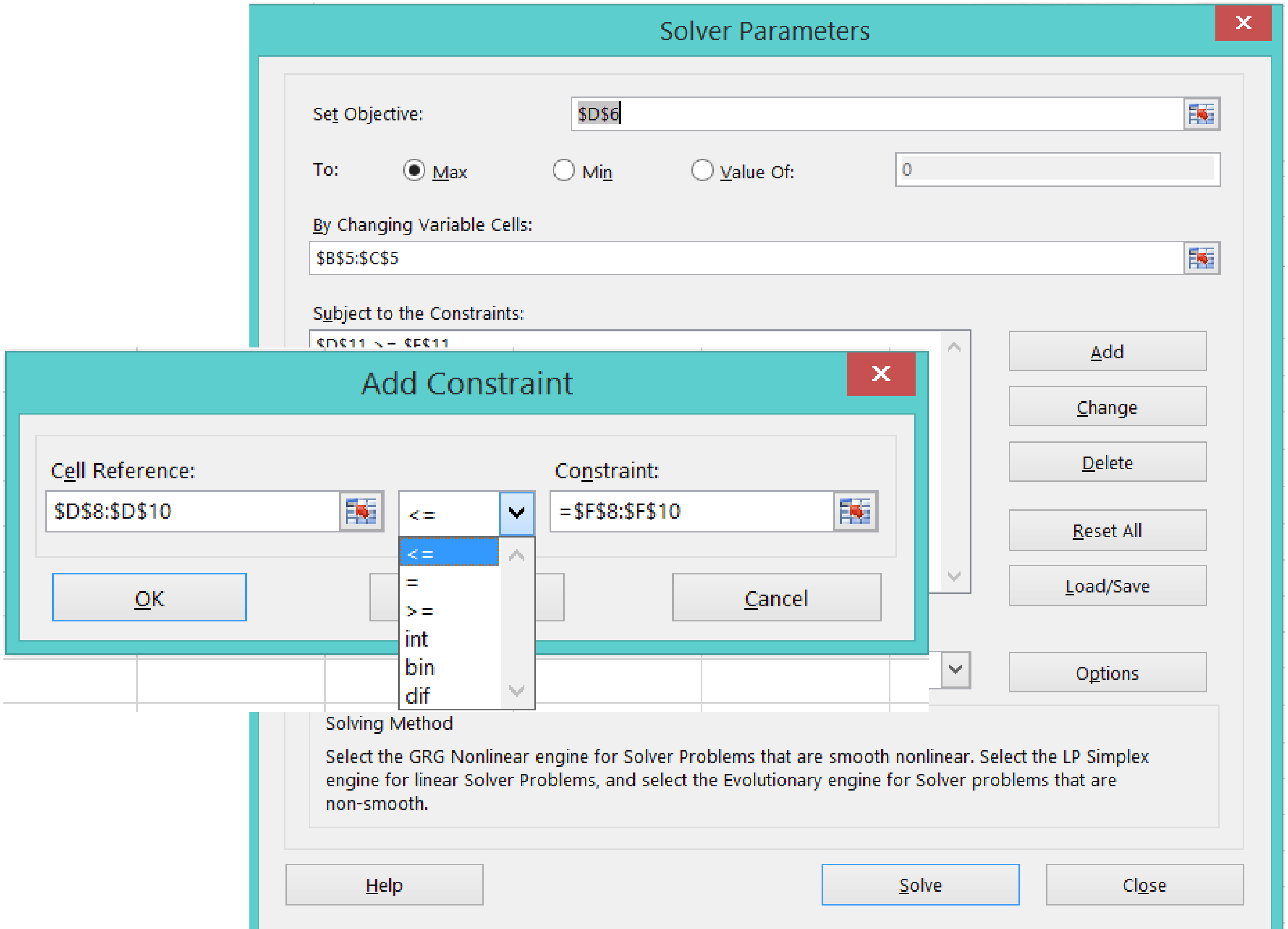


EXAMPLE EXCEL SOLUTION: FLAIR FURNITURE

Use Solver from Microsoft Excel

	A	B	C	D	E	F
1	**Flair Furniture**					
2						
3		T	C			
4		Tables	Chairs			
5	Number of Units			**Total Profit =**		
6	Profit	7	5	=SUMPRODUCT(B6:C6,B$5:C$5)		
7	**Constraints**					
8	Carpentry hours	3	4	=SUMPRODUCT(B8:C8,B$5:C$5)	<=	2400
9	Painting hours	2	1	=SUMPRODUCT(B9:C9,B$5:C$5)	<=	1000
10	Maximum chairs		1	=SUMPRODUCT(B10:C10,B$5:C$5)	<=	450
11	Minimum tables	1		=SUMPRODUCT(B11:C11,B$5:C$5)	>=	100
12				LHS	Sign	RHS

EXAMPLE EXCEL SOLUTION: FLAIR FURNITURE



EXAMPLE EXCEL SOLUTION: FLAIR FURNITURE

	A	B	C	D	E	F
1	**Flair Furniture**					
2						
3		T	C			
4		Tables	Chairs			
5	Number of Units	320	360	**Total Profit =**		
6	Profit	7	5	4040		
7	**Constraints**					
8	Carpentry hours	3	4	2400	<=	2400
9	Painting hours	2	1	1000	<=	1000
10	Maximum chairs		1	360	<=	450
11	Minimum tables	1		320	>=	100
12				LHS	Sign	RHS

STEP 3: INTERPRETATION OF RESULTS

What is the model telling you to do?

What actions should you take to:

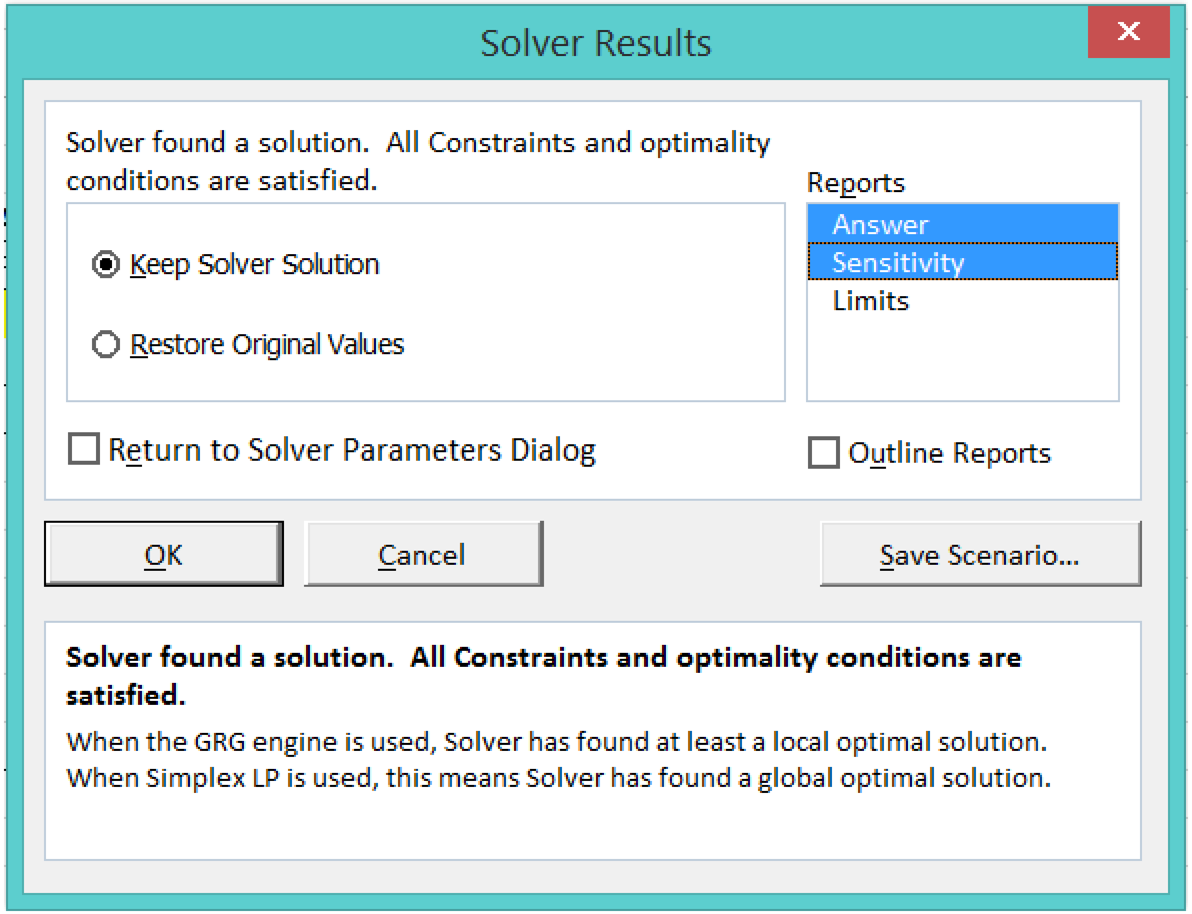
- Minimize your costs? Maximize your profits?
- Read and interpret the *MS Excel Answer Report*

Why does this make sense?

Sensitivity and economic analysis

- How is the optimal solution affected by changes and estimation errors in the problem data?
- Read and interpret the *MS Excel Sensitivity Report*

ANSWER AND SENSITIVITY REPORTS



Flair Furniture: Answer Report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$6	Profit	0	4040

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$5	Number of Units Tables	0	320	Contin
$C$5	Number of Units Chairs	0	360	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$11	Minimum tables	320	$D$11>=$F$11	Not Binding	220
$D$8	Carpentry hours	2400	$D$8<=$F$8	Binding	0
$D$9	Painting hours	1000	$D$9<=$F$9	Binding	0
$D$10	Maximum chairs	360	$D$10<=$F$10	Not Binding	90

SHADOW PRICES

Each constraint has an associated **shadow price**

The shadow price is the change in the optimal objective value per unit increase in the right-hand side of the constraint, **given that all other data remain the same**

Flair Furniture: Sensitivity Report

Microsoft Excel 15.0 Sensitivity Report
Worksheet: [Solver Example - Flair Furniture.xlsx]Sheet1
Report Created: 3/22/2015 5:05:37 PM

Range Information

Shadow Price Information

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number of Units Tables	320	0	7	3	3.25
$C$5	Number of Units Chairs	360	0	5	4.333333333	1.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$11	Minimum tables	320	0	100	220	1E+30
$D$8	Carpentry hours	2400	0.6	2400	225	900
$D$9	Painting hours	1000	2.6	1000	600	150
$D$10	Maximum chairs	360	0	450	1E+30	90

SENSITIVITY ANALYSIS QUESTIONS

- Flair Furniture is offered 100 more painting hours at a cost of $250. Should they take the deal?
- Flair Furniture is offered 100 more carpentry hours at a cost of $250. Should they take the deal?
- What would be the impact of decreasing the minimum number of tables in 50 units?
- What would be the impact of increasing the maximum number of chairs in 50 units?

GENERAL PRINCIPLES ON SHADOW PRICES

- The unit of the shadow price is the unit of the objective function divided by the unit of the constraint

$$\text{Shadow Price} = \frac{\Delta\ \text{(optimal objective function value)}}{\Delta\text{(RHS value)}}$$

- In terms of microeconomic theory, the shadow price of a given constraint is the marginal value of the resource whose units are expressed in the constraint
- Shadow prices values are valid in a range. Outside of that range it is necessary to resolve the LP
- The shadow price for any non-binding constraint will be zero

MEDIA SELECTION PROBLEM (MSP)

Kitchener Electronics:

- Four advertising media:

TV spots, newspaper ads, and two types of radio advertisements

- Budget: $8,000 per week for advertising

Decision: How many ads of each type?

Objective: Maximize audience exposure

	Advertising Options			
	TV Spot	**Newspaper**	**Radio (prime time)**	**Radio (afternoon)**
Audience Reached (per ad)	5000	8500	2400	2800
Cost (per ad)	$800	$925	$290	$380
Max Ads (per week)	12	5	25	20

STEP 1: LP FORMULATION OF MSP DECISION VARIABLES

Other Restrictions

Have at least 5 radio spots / week

Spend no more than $1800 on radio ads / week

Decision Variables

T = number of TV spots per week

N = number of newspaper ads per week

P = number of prime time radio spots per week

A = number of afternoon radio spots per week

STEP 1: LP FORMULATION OF MSP OBJECTIVE &CONSTRAINTS

Objective Function (audience reached):

Max $Z = 5000T + 8500N + 2400P + 2800A$

Subject to the constraints:

$800T + 925N + 290P + 380A < 8000$ (Budget)

$P + A > 5$ (minimum radio spots per week)

$290P + 380A \leq 1800$ (Maximum \$ on radio)

$T \leq 12$ $P \leq 25$

$N \leq 5$ $A \leq 20$ (maximum number of ads per week for each type)

$T, N, P, A \geq 0$ (non-negativity)

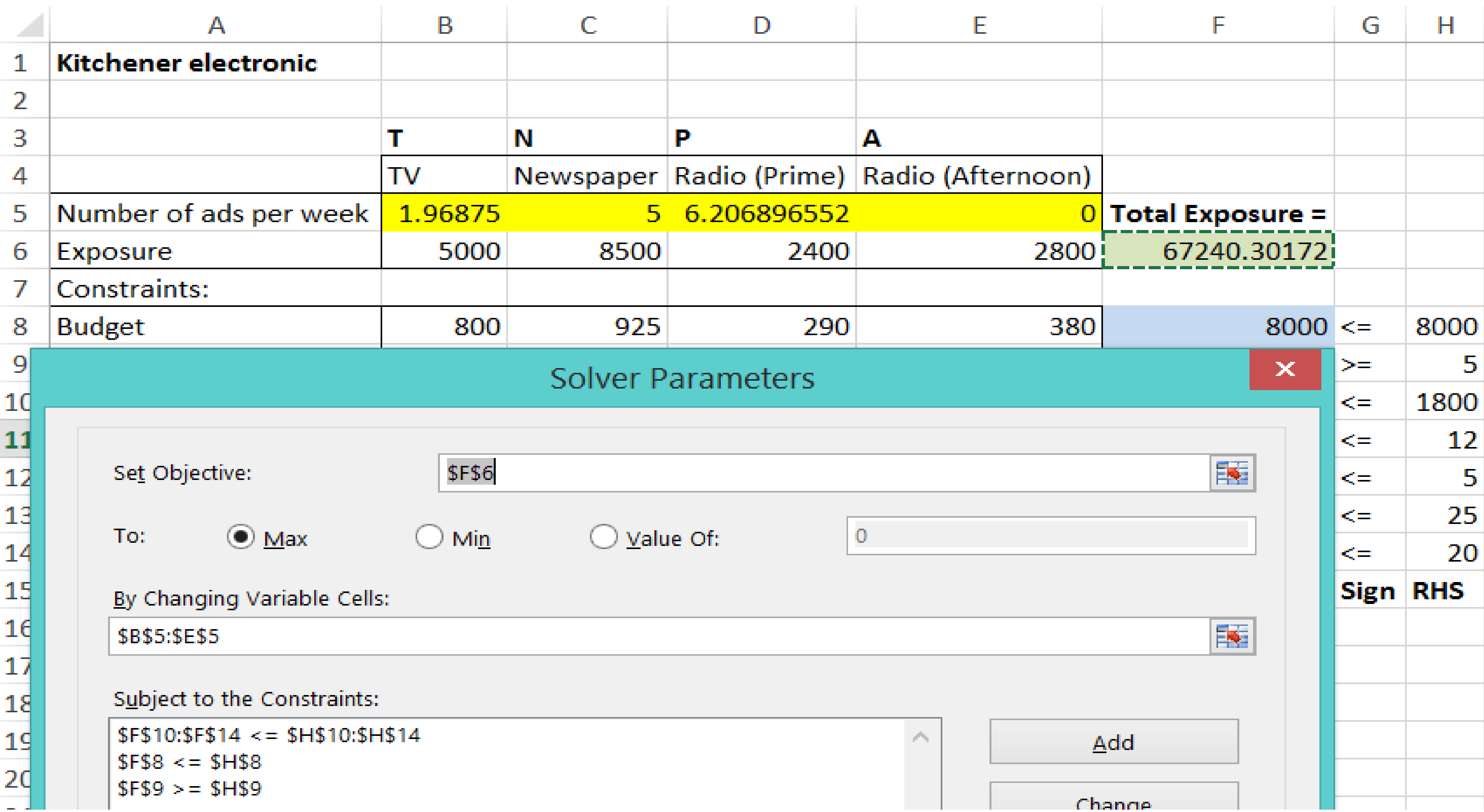
STEP 2: SOLUTION TECHNIQUE

Use Solver from Microsoft Excel

	A	B	C	D	E	F	G	H
1	**Kitchener electronic**							
2								
3		**T**	**N**	**P**	**A**			
4		TV	Newspaper	Radio (Prime)	Radio (Afternoon)			
5	Number of ads per week					**Total Exposure =**		
6	Exposure	5000	8500	2400	2800	=SUMPRODUCT(B6:E6,B$5:E$5)		
7	Constraints:							
8	Budget	800	925	290	380	=SUMPRODUCT(B8:E8,B$5:E$5)	<=	8000
9	Min radio spots			1	1	=SUMPRODUCT(B9:E9,B$5:E$5)	>=	5
10	Max $ on radio			290	380	=SUMPRODUCT(B10:E10,B$5:E$5)	<=	1800
11	TV max ads per week	1				=SUMPRODUCT(B11:E11,B$5:E$5)	<=	12
12	N max ads per week		1			=SUMPRODUCT(B12:E12,B$5:E$5)	<=	5
13	RP max ads per week			1		=SUMPRODUCT(B13:E13,B$5:E$5)	<=	25
14	RA max ads per week				1	=SUMPRODUCT(B14:E14,B$5:E$5)	<=	20
15						**LHS**	**Sign**	**RHS**

STEP 2: SOLUTION TECHNIQUE

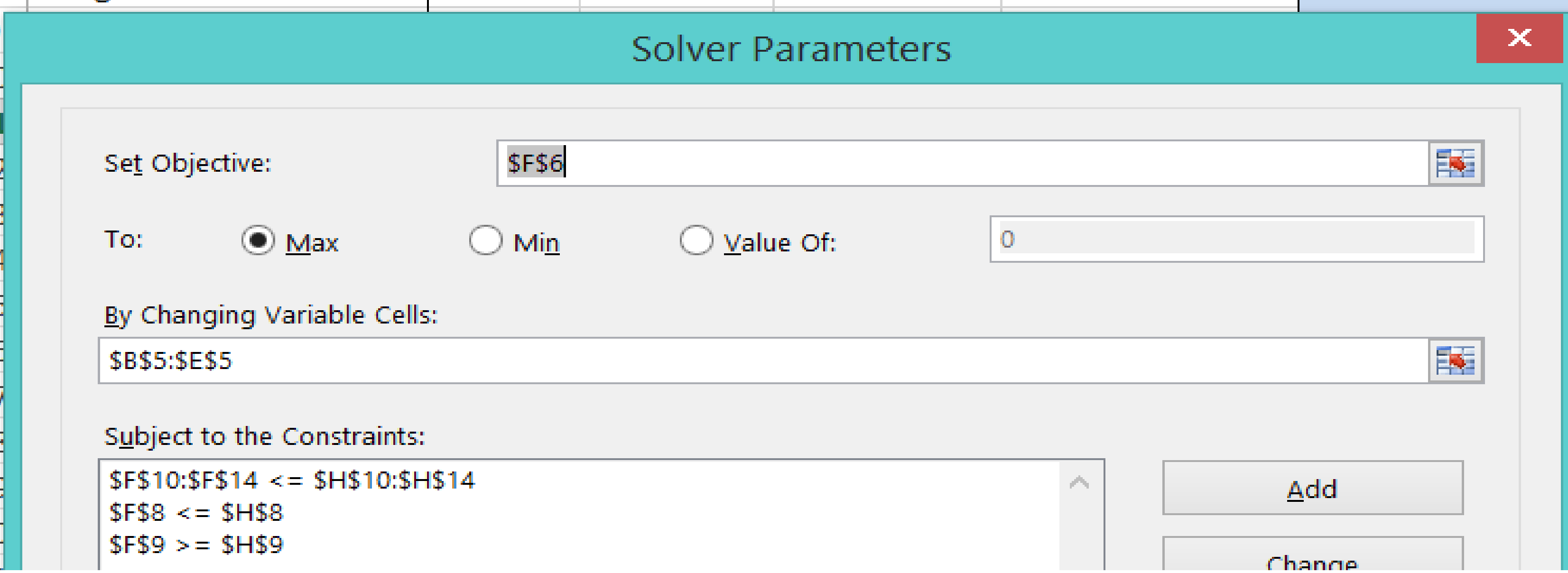
Use Solver from Microsoft Excel



STEP 2: SOLUTION TECHNIQUE

Use Solver from Microsoft Excel

	A	B	C	D	E	F	G	H
1	**Kitchener electronic**							
2								
3		**T**	**N**	**P**	**A**			
4		TV	Newspaper	Radio (Prime)	Radio (Afternoon)			
5	Number of ads per week	1.96875	5	6.206896552	0	**Total Exposure =**		
6	Exposure	5000	8500	2400	2800	67240.30172		
7	Constraints:							
8	Budget	800	925	290	380	8000	<=	8000
9							>=	5
10							<=	1800
11							<=	12
12							<=	5
13							<=	25
14							<=	20
15							**Sign**	**RHS**



Objective Cell (Max)

Cell	Name	Original Value	Final Value
$F$6	Exposure	0	67240.30172

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$5	Number of ads per week TV	0	1.96875	Contin
$C$5	Number of ads per week Newspaper	0	5	Contin
$D$5	Number of ads per week Radio (Prime)	0	6.206896552	Contin
$E$5	Number of ads per week Radio (Afternoon)	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$10	Max $ on radio	1800	$F$10<=$H$10	Binding	
$F$11	TV max ads per week	1.96875	$F$11<=$H$11	Not Binding	10.0312
$F$12	N max ads per week	5	$F$12<=$H$12	Binding	
$F$13	RP max ads per week	6.206896552	$F$13<=$H$13	Not Binding	18.7931034
$F$14	RA max ads per week	0	$F$14<=$H$14	Not Binding	2
$F$8	Budget	8000	$F$8<=$H$8	Binding	
$F$9	Min radio spots	6.206896552	$F$9>=$H$9	Not Binding	1.20689655

Shadow Price Information **Range Information**

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number of ads per week TV	1.96875	0	5000	1620.689655	5000
$C$5	Number of ads per week Newspaper	5	0	8500	1E+30	2718.75
$D$5	Number of ads per week Radio (Prime)	6.206896552	0	2400	1E+30	263.1578947
$E$5	Number of ads per week Radio (Afternoon)	0	-344.8275862	2800	344.8275862	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$10	Max $ on radio	1800	2.025862069	1800	1575	350
$F$11	TV max ads per week	1.96875	0	12	1E+30	10.03125
$F$12	N max ads per week	5	2718.75	5	1.702702703	5
$F$13	RP max ads per week	6.206896552	0	25	1E+30	18.79310345
$F$14	RA max ads per week	0	0	20	1E+30	20
$F$8	Budget	8000	6.25	8000	8025	1575
$F$9	Min radio spots	6.206896552	0	5	1.206896552	1E+30

SENSITIVITY ANALYSIS QUESTIONS

- **How would the audience exposure change if the maximum number of Newspaper ads was increased by one?**
- **Would you rather have $320 of extra budget, or have the same budget but increase the expense limit on radio ads per week to $2800?**

EXTENSIONS TO LINEAR PROGRAMMING

(Mixed) Integer Programming

- (Some of) the decision variables must be an integer

Non-Linear Programming

- Convex programming
  - Objective function is a convex function
  - Constraints define a convex set
- Nonconvex optimization
  - Global optimization, approximate methods

Dynamic Programming

SUMMARY: LINEAR PROGRAMMING

- **Define the decision variables**
- **Formulate LP using the decision variables**
  - Write the objective function equation
  - Write each of the constraints
- **Implement the model (e.g., in Excel)**
- **Solve (e.g., graphically or Excel's Solver)**
- **Interpret your results and do sensitivity analysis**